Kirby calculus

Gompf, Stipsicz "4-Manifolds and Kirby Calculus"

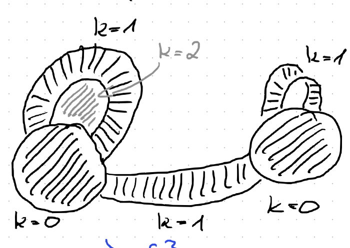
1. Handle decompositions

M smooth n-Mf, $\varphi : \partial D^k \times D^{n-k} \to \partial M$ smooth emb.

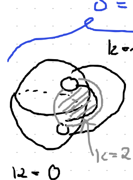
$M \cup_\varphi h_k := M \cup_\varphi D^k \times D^{n-k}$

Examples

$n=2$



$n=3$



Theorem

Every smooth connected closed n-manifold admits a handle decomposition

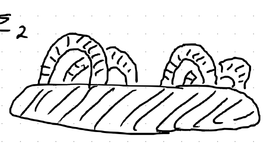
$M \cong \emptyset \cup_{\varphi_0} h_0 \cup_{\varphi_1} h_1 \cup \cdots \cup h_1$
$\vdots$
$\cup h_{n-1} \cdots \cup h_{n-1} \cup h_n$

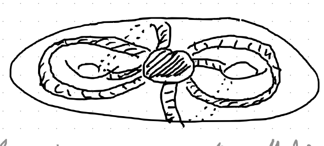
with precisely one 0-handle and one n-handle. Furthermore, k-handles are only attached to $\underbrace{\bigcup_{i<k} i\text{-handles}}_{k-1 \text{ skeleton}}$

Examples

- $S^n = h_0 \cup_{id} h_n$ ($D^0 \times D^n$, $D^n \times D^0$)

- Σ_2

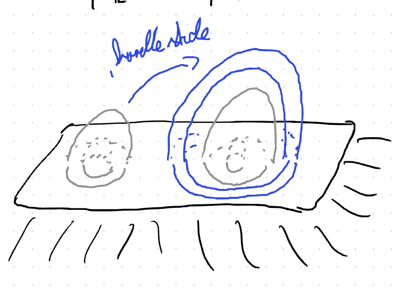




„Problem: Imprecision regarding attaching maps" in particular for h_n

Prop

$\varphi, \psi : \partial D^k \times D^n \hookrightarrow \partial M$ isotopic, smooth emb.
$\Rightarrow M \cup_\varphi h_k \cong M \cup_\psi h_k$



Prop

$n \leq 4$: $\varphi, \psi : \partial D^n \times D^0 \hookrightarrow \partial M$ map to same component
$\Rightarrow M \cup_\varphi h_n \cong M \cup_\psi h_n$

Proof

$\{\text{self-diffeos of } S^n\}/\text{isotopy} = \{id, \text{reflection}\}$ — both extend over D^n

Smale-Milnor, Cerf ≈60

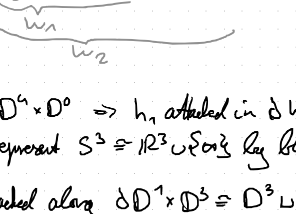
2. Kirby diagrams

M closed, or., smooth, connected 4-manifold with a handle decomposition:

$M = \underbrace{\underbrace{\underbrace{h_0}_{W_0} \cup h_1 \cup \cdots \cup h_1}_{W_1} \cup h_2 \cdots h_2}_{W_2} \cup h_3 \cdots h_3 \cup h_4$

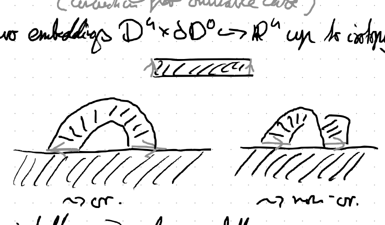
$W_0 = D^4 \times D^0 \Rightarrow h_1$ attached in $\partial W_0 \cong S^3$
$\rightsquigarrow$ represent $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ by blackboard

h_1 attached along $\partial D^1 \times D^3 \cong D^3 \sqcup D^3$



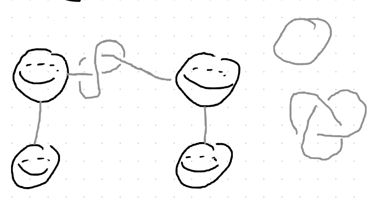
identify by a reflection
(careful in non-orientable case)

two embeddings $D^4 \times \partial D^0 \hookrightarrow \mathbb{R}^4$ up to isotopy

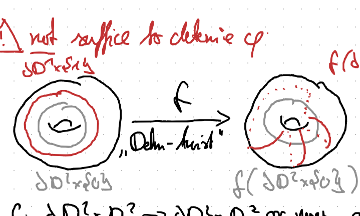


M orientable $\Rightarrow$ only one of them appears

h_2 attached along
$\varphi : \partial D^2 \times D^2 \hookrightarrow \partial W_1$
draw attaching sphere $\varphi(\partial D^2 \times \{0\}) \subseteq \partial W_1$ as knot:



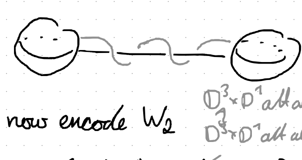
⚠ not suffice to determine φ



$f : \partial D^2 \times D^2 \to \partial D^2 \times D^2$ or. pres. diffeo
not isotopic to id

up to isotopy all or. pres. self-diffeos of $\partial D^2 \times D^2$ are such Dehn-twists

$\Rightarrow$ to determine $\varphi : \partial D^2 \times D^2 \hookrightarrow \mathbb{R}^4$ suffices:
$\varphi(\partial D^2 \times \{0\})$ knot
$\varphi(\partial D^2 \times \{1\})$ framing



can now encode W_2

$D^3 \times D^1$ attached along $\partial D^3 \times D^1$ to W_2
$D^3 \times D^1$ attached along $D^3 \times \partial D^1$ to h_4
and $h_4 \cong h_0$

If M is closed, this determines M:
$M \setminus \mathring{W}_2 = h_3 \cup \cdots \cup h_3 \cup h_4$
by handle-duality can be written as
$M \setminus \mathring{W}_2 \cong \underbrace{h_0 \cup h_1^1 \cup \cdots \cup h_1^k}_{=: X_k}$

$\Rightarrow M = W_2 \cup_\varphi X_k$ for some diffeo $X_k \xrightarrow{\varphi} \partial W_2$

$\partial W_2 \cong \partial X_k = \#^k S^1 \times S^2$
$\Rightarrow k$ is determined by W_2

Theorem (Laudenbach, Poenaru ≈ 70s)
Any self diffeomorphism of ∂X_k extends over X_k.

Examples

- empty diagram:
  $D_4 = W_0 = W_1 = W_2$
  $\Rightarrow k=0$
  $M = h_0 \cup h_4 \cong S^4$

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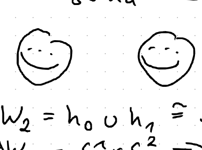

$W_2 = h_0 \cup h_1 \cong S^1 \times D^3$
$\partial W_2 = S^1 \times S^2 \Rightarrow k=1$

$M = S^1 \times D^3 \cup S^1 \times D^3 = S^1 \times S^3$

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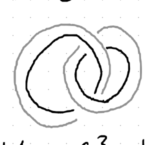

$W = S^2 \times S^2$

Idea: $S^2 = h_0 \cup h_2 = D^2_- \times D^2_+$
$\rightsquigarrow S^2 \times S^2 = \underbrace{D^2_- \times D^2_-}_{h_0} \cup \underbrace{D^2_- \times D^2_+}_{h_2} \cup \underbrace{D^2_+ \times D^2_-}_{h_2} \cup \underbrace{D^2_+ \times D^2_+}_{h_4}$

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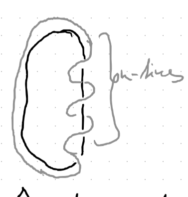


$\partial W_2 \cong L(n,1)$
$\begin{cases} \cong S^1 \times S^2 & n=0 \\ \cong S^3 & n=\pm 1 \\ \not\cong \#^k S^1 \times S^2 & \text{else} \end{cases}$ (π_1)

⚠ not every diagram gives a closed manifold, some only give W_2

$n=0$: $S^4 = S^2 \times D^2 \cup D^2 \times S^2$
$n=\pm 1$ $\mathbb{C}P^1$

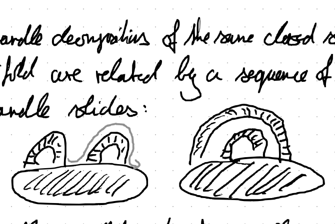
Framings of a knot $\simeq \mathbb{Z}$
$\rightsquigarrow$ choice of 0 if knot $\subseteq S^3$ by linking number
(or bounds a surface in ∂W_1)

3. Kirby calculus

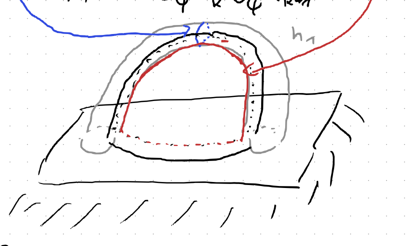
Theorem

Two handle decompositions of the same closed smooth n-manifold are related by a sequence of

- handle slides:

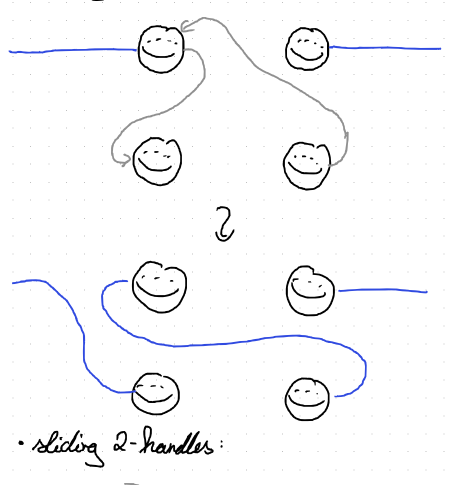


- handle cancellations / creating cancelling pairs:
  $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial M$
  $\psi : \partial D^{k+1} \times D^{n-k-1} \hookrightarrow \partial M \cup_\varphi h^k$
  st. „belt sphere" $\{0\} \times \partial D^{n-k} \pitchfork \psi(\partial D^{k+1} \times \{0\})$ „attaching sphere"
  in one point. Then
  $M \cong M \cup_\varphi h_k \cup_\psi h_{k+1}$

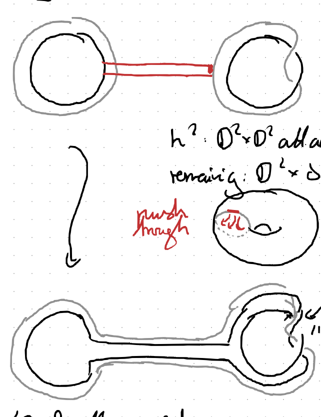


Kirby calculus:

- sliding 1-handles:



- sliding 2-handles:

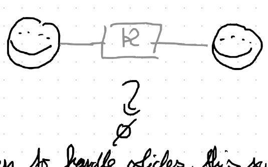


h^2: $D^2 \times D^2$ attached along $\partial D^2 \times D^2$
remaining: $D^2 \times \partial D^2$

push through

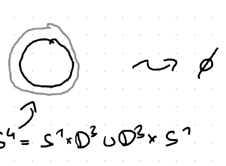
„framing gets push through first"

- 1/2-handle cancelation



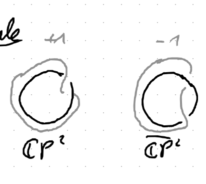
up to handle slides, this suffice
Observe: need to push other 2-handles of the 1-handle first

- 2/3-handle cancellation

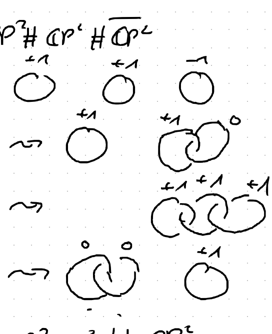


$S^4 = S^1 \times D^3 \cup D^3 \times S^1$

Example



$\mathbb{C}P^2 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$



$S^2 \times S^2 \# \mathbb{C}P^2$

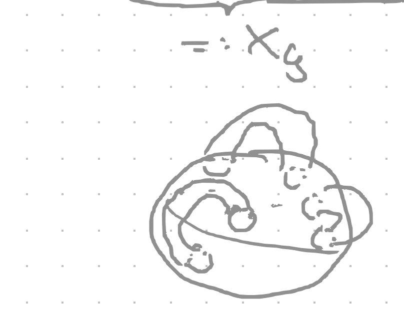
Trisections

Gay, Kirby „Trisecting 4-manifolds"

1. Heegaard splittings

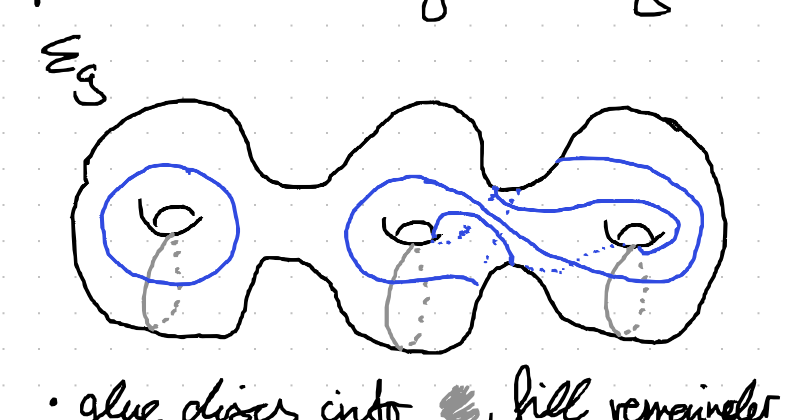
M closed, connected, oriented, smooth, 3-dim Mf

$$M = \underbrace{h_0 \cup h_1^1 \cup \dots \cup h_1^g}_{=: X_g} \cup \underbrace{h_2^1 \cup \dots \cup h_2^{g'} \cup h_3}_{= X_{g'}}$$



$$\partial X_g = \Sigma_g \overset{!}{=} \Sigma_{g'} = \partial X_{g'}$$
$$\Rightarrow g = g'$$

represented in a Heegaard diagramm:

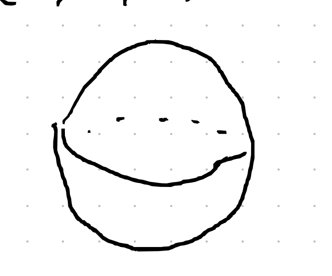


- glue discs into, fill remainder with B^3
  $\rightsquigarrow$ obtain X_g
- same procedure on
  $\rightsquigarrow$ obtain X_g

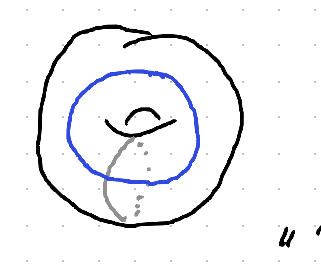
Def (Heegaard diagramm)
A Heegaard diagramm is a triple (g, α, β) where α and β are sets of g disjoint curves in Σ_g. (can take α to be the standard curves)

Examples

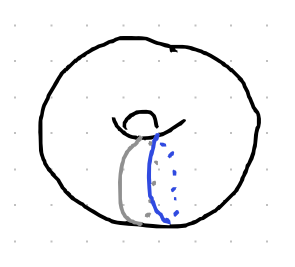
- $(0, \emptyset, \emptyset)$



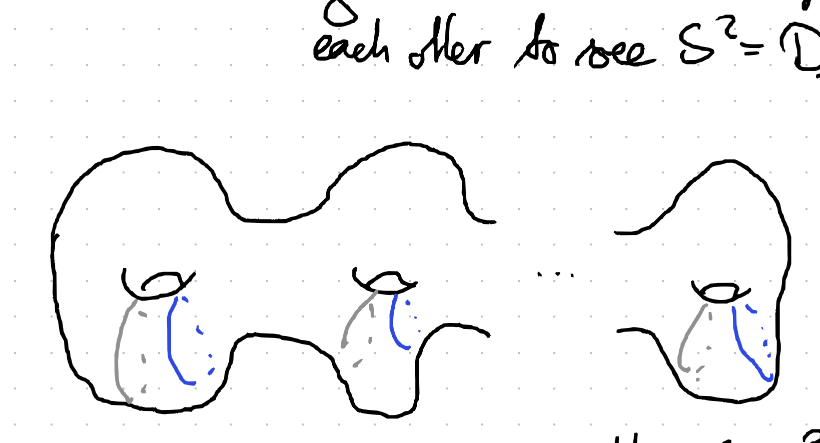
$\rightsquigarrow S^3$

- 

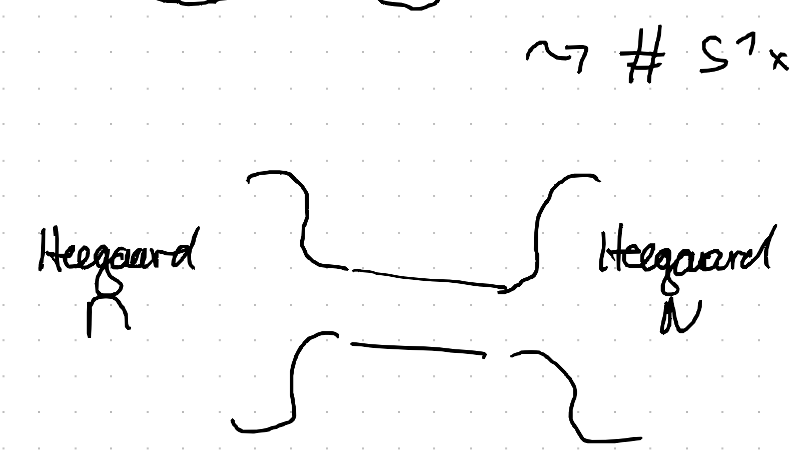
$\rightsquigarrow S^3$
„two tori glued together"
$T(p,q) \rightsquigarrow L(p,q)$

- 

$\rightsquigarrow S^1 \times S^2$
„glue the two discs on top of each other to see $S^2 = D^2_- \cup D^2_+$"

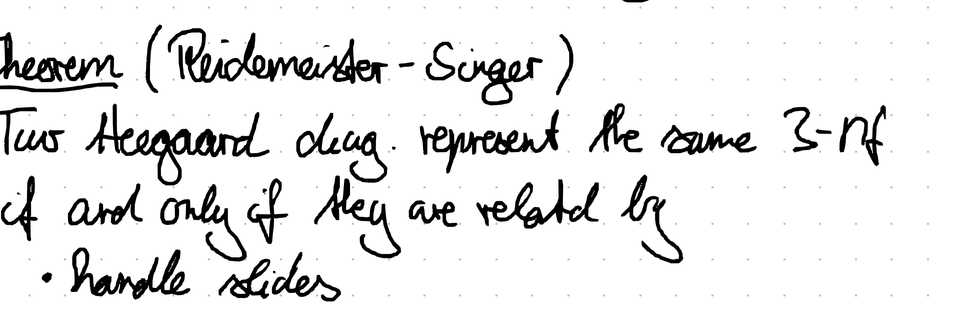


$\rightsquigarrow \# S^1 \times S^2$

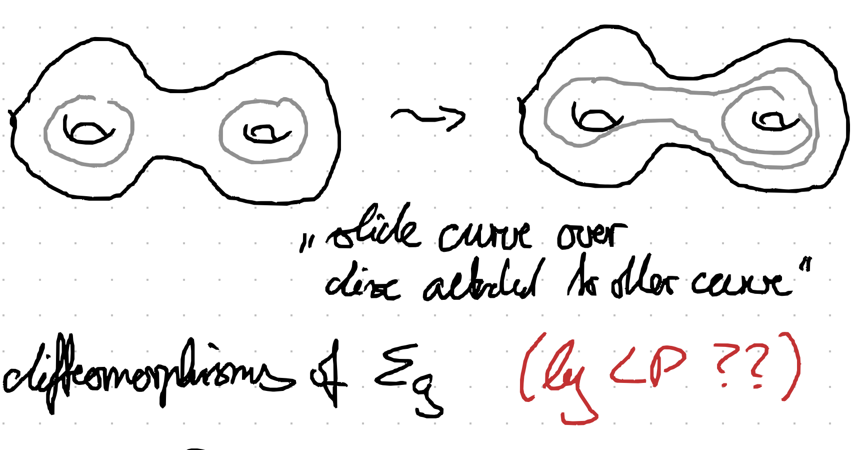
- 

$\rightsquigarrow M \# N$
„take out $B^3 \subseteq M, N$ st it intersects the surface in a disc not touching the curves"

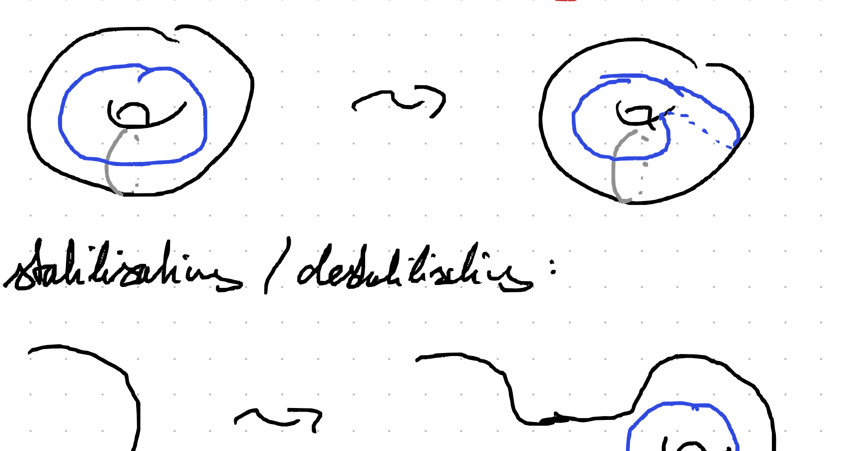
Theorem (Reidemeister-Singer)
Two Heegaard diag. represent the same 3-Mf if and only if they are related by
- handle slides



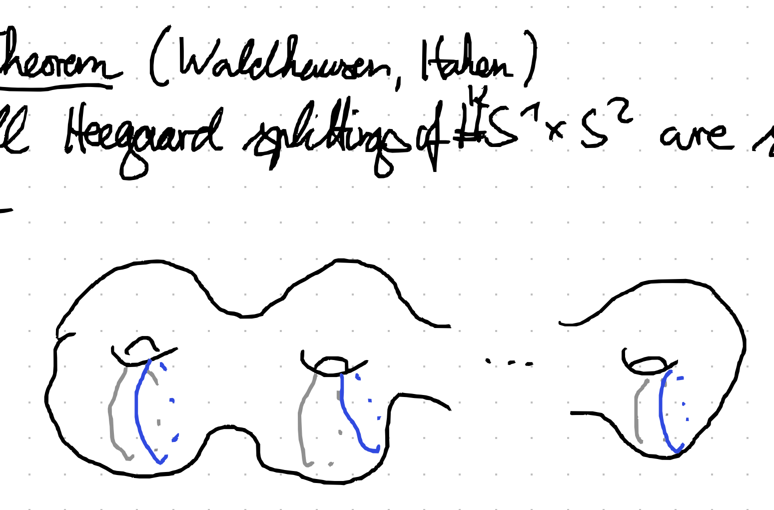
„slide curve over disc attached to other curve"
- diffeomorphisms of Σ_g (by LP ??)



- stabilization / destabilisation:



Theorem (Waldhausen, Haken)
All Heegaard splittings of $\#^k S^1 \times S^2$ are stabilisation of



Remark
Heegaard splittings describe a handle decomp:
- the first set of curves describes belt spheres 1-handles
- the second set describes attaching spheres of 2-handles

Stabilization then is a 1/2-handle cancellation.

$\rightsquigarrow$ Analogue to Kirby calculus in 3-dim
Question: Kirby calculus in 2-dim?

Remark (Morse theoretic viewpoint)

$M \longrightarrow \mathbb{R}$ self-indexing Morse function

3 ← only one
2
1.5 „Heegard surface"
1
0 ← only once

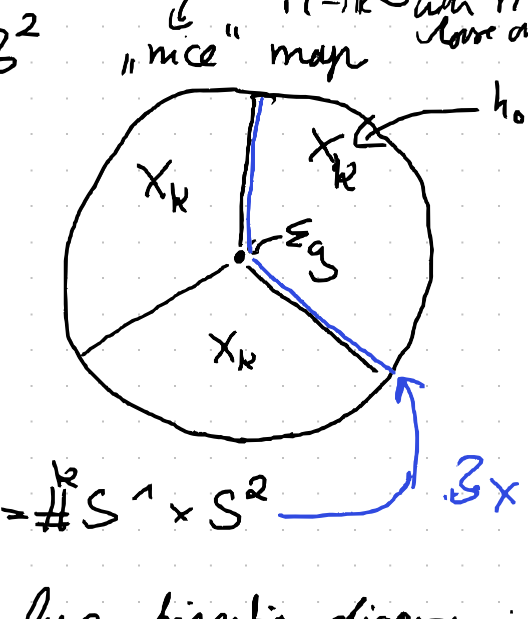
$\}\ X_{g'}$ (levels 2–3), $\}\ X_g$ (levels 0–1)

2. Trisections

M closed, orientable, connected smooth 4-manifold

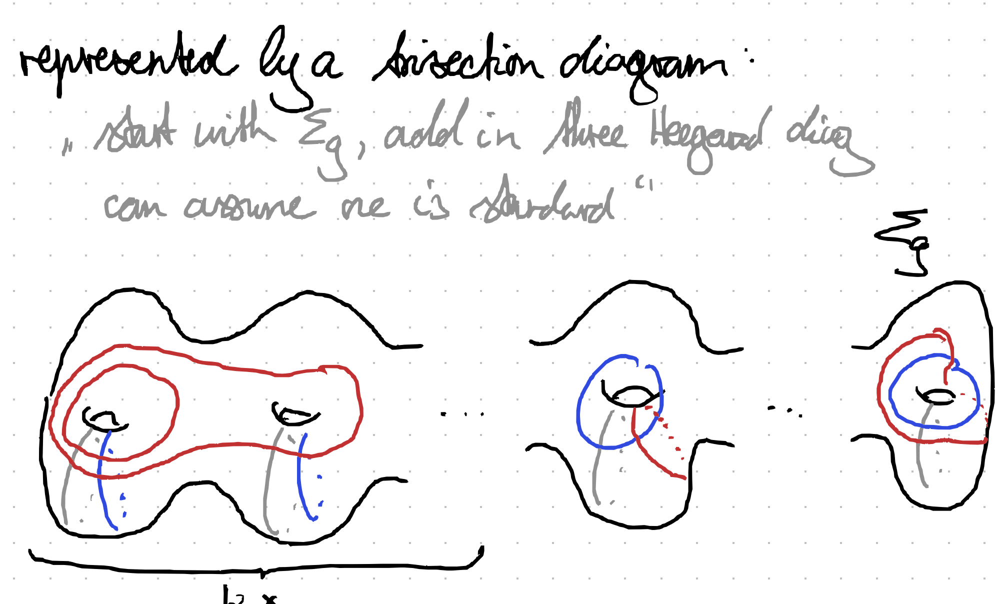
„Increase dim by 1. As drawn preimage of pt must be Σ_g
$\rightsquigarrow M \longrightarrow \mathbb{R}^2$"

$M \longrightarrow B^2$ „nice" map — essentially 2-Morse, i.e. $M \to \mathbb{R}^2$ with $M \to \mathbb{R}$ Morse and Morse on fibers



$\partial X_k = \#^k S^1 \times S^2$ — $3\times$

represented by a trisection diagram:
„start with Σ_g, add in three Heegaard diag can assume one is standard"



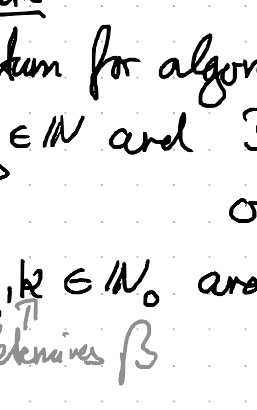
„This denotes how to attach, suffices by LBP"

Def (trisection diagram)
A trisection diagram is a tuple $(g, \alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \subseteq \Sigma_g$ are three sets of disjoint curves

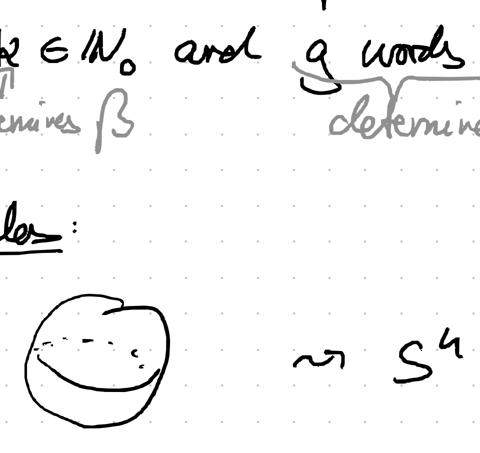
Theorem
A trisection diagram $(g, \alpha, \beta, \gamma)$ gives a unique clo. or., con. smooth 4-dim. Mf if and only if $(g, \alpha, \beta), (g, \alpha, \gamma), (g, \beta, \gamma)$ are Heegaard diagrams for $\# S^1 \times S^2$.

Remark
Datum for algorithm is clear:
$g \in \mathbb{N}$ and $3\times$ g words in F_g
or (if standard)
$g, k \in \mathbb{N}_0$ and $\underbrace{g \text{ words in } F_g}_{\text{determines } \gamma}$
(k determines β)

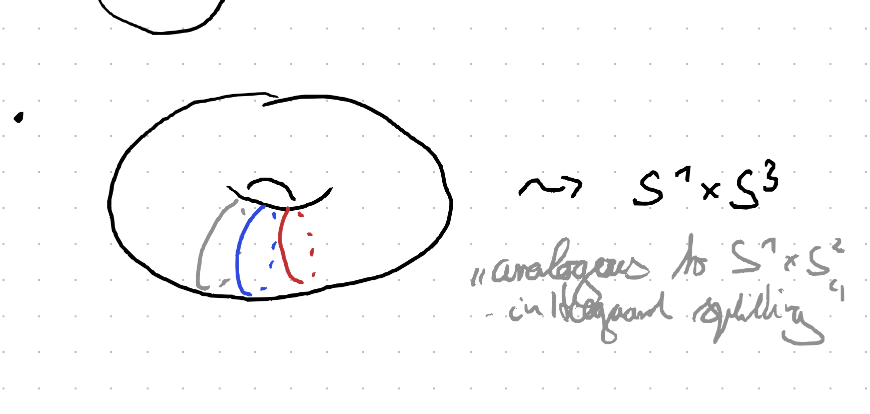
Examples:

- 

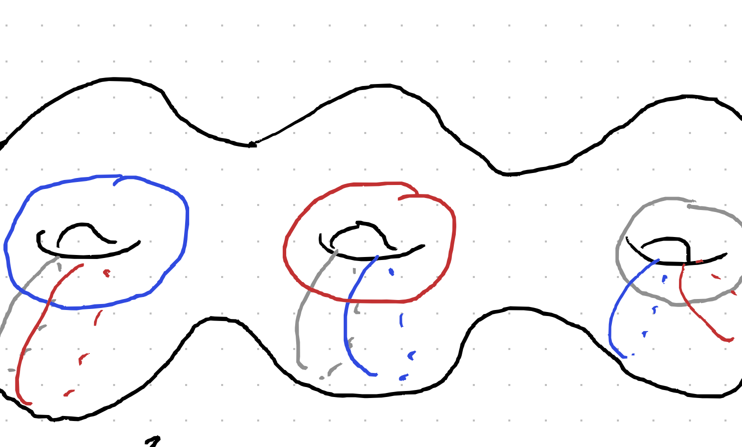
$\rightsquigarrow S^4$

- 

$\rightsquigarrow S^1 \times S^3$
„analogous to $S^1 \times S^2$ in Heegaard splitting"

- 

$\rightsquigarrow S^3$

- 

$\rightsquigarrow M \# N$

Theorem
Two trisection diag. give diffeomorphic 4-Mf if and only if they are related by
- diffeos of Σ_g
- handle slides (as in Heegaard diag)
- stabilizations

Examples
$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \cong S^2 \tilde{\times} S^2$

Triangulations

Friedl Chapter: Simplicial complexes

1. PL-manifolds

Def (abstract simplicial complex)

vertices, simplices

An abstract simplicial complex is a pair $K=(V,S)$ where V is a set and $S \subseteq \mathcal{P}(V)$ st.:

- $\forall s \in S$: s is finite, non-empty
- $\forall s \in S,\ s' \subseteq s$: $s' \in S$
- $\forall s \in S,\ v \in s$: $v \in V$

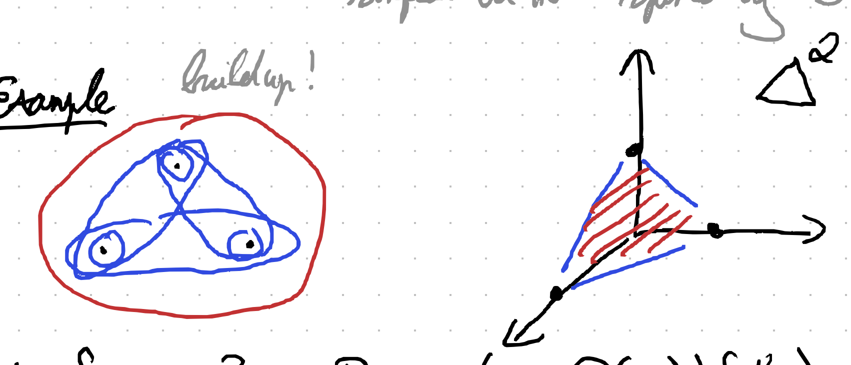
The top. realization is the subset of $\mathbb{R}^{(V)}$

$V \subseteq \mathbb{R}^{(V)}$ by $v \mapsto e_v$

$$|K| = \bigcup_{s \in S} |s|$$

$|s|$ = simplex in $\mathbb{R}^{(V)}$ spanned by s

Example buildup!



$V := \{0, \dots, n\}$: $D_n := (V, \mathcal{P}(V) \setminus \{\emptyset\})$

$S_{n-1} := (V, \mathcal{P}(V) \setminus \{\emptyset, V\})$

Def (simplicial complex)

Let X top. space. A simp. structure on X is a pair (K, Φ) where

- K is a abstract simp. complex
- $\Phi: X \to |K|$ is a homeo.

A sim. complex is a top. space with a simp. structure.

Def (subdivision/PL)

A simplicial complex A is a subdivision of a simp. complex X if:

- the top. spaces of A, X are the same
- all simplices in A are contained in simplices of X
- $\forall t$ n-simplex in A, s m-simplex in X containing t

$$\Phi_s^{-1} \circ \Phi_t : \Delta^n \to \Delta^m \text{ affine linear}$$

$$(\Delta^n \xrightarrow{\cong} |t| \subseteq |A|)$$

A map $f: X \to Y$ (between simplicial complexes) is piecewise linear, if there exist subdivisions X', Y' st. $f: X' \to Y'$ is simplicial. ⚠ composition only clearly PL if X, Y finite

A map $f: X \to Y$ between simp. complexes is a PL-homeomorphism, iff f is bij. and f, f^{-1} PL.

Def (PL-manifold)

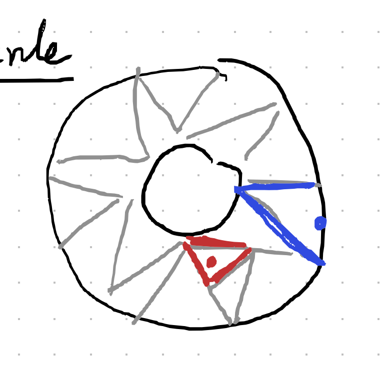
neighborhood = subcomplex

A PL-manifold is a simp. complex with countably many simplices st. every point has a neighborhood PL-homeomorphic to Δ^n.

A PL-manifold X is closed if X is compact and for every vertex v of X the link of v is a PL-sphere

$\{s \in S \mid v \notin s,\ s \cup \{v\} \in S\}$

Example ⚠ can have boundary



Theorem (smooth vs. PL)

The underlying top. space of every smooth Mf admits a PL-structure:

$$\left\{ \begin{matrix} \text{diffeo. classes of} \\ \text{closed smooth } n\text{-Mf} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{PL-homeo classes of} \\ \text{closed PL-Mf} \end{matrix} \right\}$$

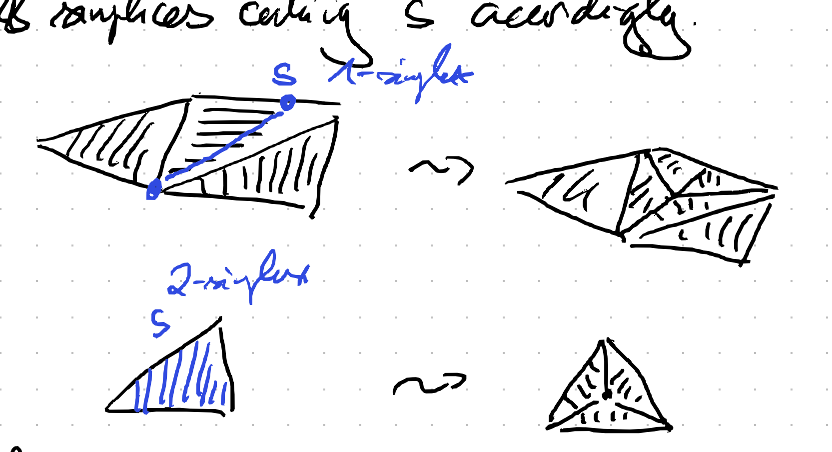
$$M \longmapsto M$$

For $n=4$ this is a bijection.

2. combinatorial manifolds

Def (stellar subdivision)

Let $K=(V,S)$ abstract simplicial complex and $s \in S$. The stellar subdivision at s is given by adding a vertex in the middle of s and subdividing all simplices containing s accordingly.



The inverse operation is a stellar weld.

Theorem (Alexander-Newman 1930)

A, B abstract simplicial complexes

A, B stellar equivalent $\Longleftrightarrow$ $|A|, |B|$ PL-homeo.

Def (combinatorial manifold)

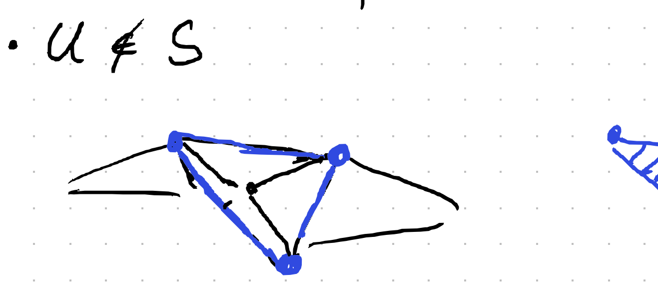
A finite abstract simplicial complex is a (closed) n-dim combinatorial manifold if the link of every vertex is stellar equivalent to S_{n-1}

Def (Pachner move)

Let (V,S) abstract simp. complex and $s \in S$

A subcomplex (U, R) is a non-trivial m-sphere if

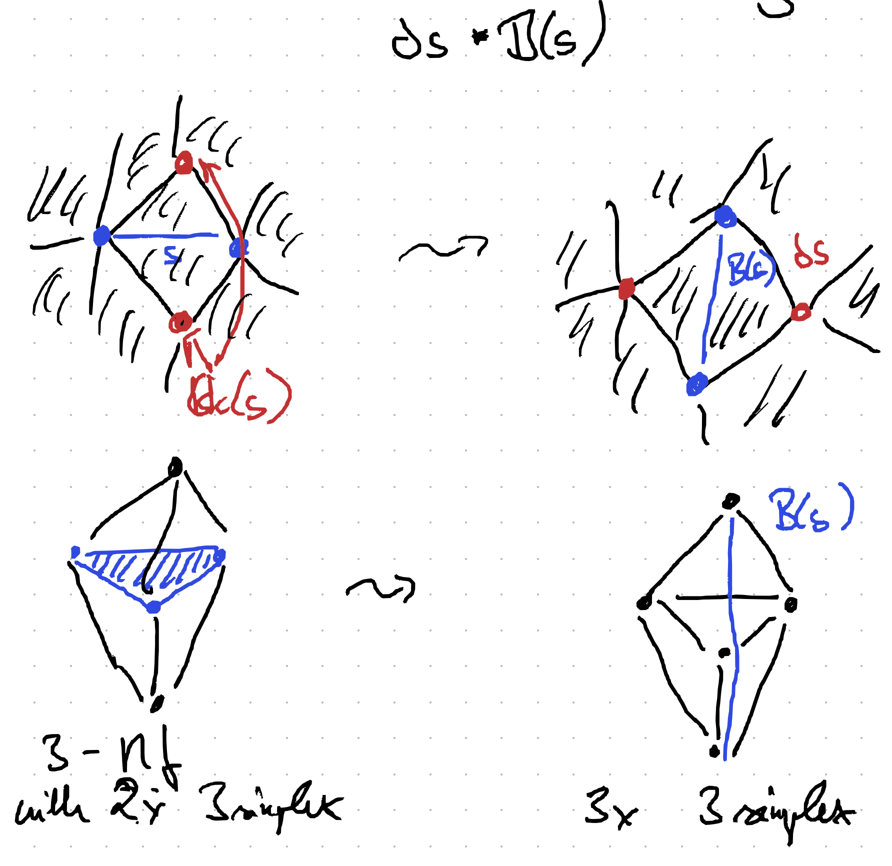
- $\#U = m+2$, $R = \mathcal{P}(U) \setminus \{\emptyset, U\}$
- $U \notin S$



If $lk(s)$ is a non-trivial m-sphere we define $B(s) := (U, \mathcal{P}(U) \setminus \emptyset)$

If (V,S) is a comb. n-mf and $lk(s)$ a non-trivial m-sphere, the bistellar/Pachner move along s replaces

$$St(s) = s * lk(s) \quad \text{by} \quad \partial s * B(s)$$



Theorem (Pachner 1991)

K, L combinatorial Mf

K, L stellar equivalent $\Longleftrightarrow$ K, L bistellar equivalent

$$\begin{array}{c} \{\text{smooth closed 4-Mf}\}/\text{diffeo} \\ \downarrow \cong \\ \{\text{closed PL-4-Mf}\}/\text{PL-homeo} \\ \downarrow \cong \\ \{\text{combinatorial Mf}\}/\text{stellar equivalence} \\ \| \\ \{\text{combinatorial Mf}\}/\text{bistellar equivalence} \end{array}$$

can be used to define invariants

Questions

- Translate between the different representations

  see Gay, Kirby (Kirby diag = framed link diag) Chapter 4

$$\begin{tikzcd}
\text{Kirby diag.} \arrow[dr] & & \text{Trisections} \arrow[ll, bend right] \arrow[dl] \\
& \text{Triangulations} &
\end{tikzcd}$$

- Calculate alg. top. invariants:

  very doable for Kirby diagrams
  (π_1, H_*, intersection form?)

  doable for trisections
  (π_1, H_* ⇝ show $g=3$ minimal, non-trivial for S^3)

- How to represent a Kirby diag. for an algo?
  PL-mf ← see Friedl, chapter 1&2.

- How to determine if a Kirby diag. gives a closed Mf?
  Trise. diag. ← 3-mf questions